acting at the interface, but it is not affected by the g-field distributions throughout the fluid masses.

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Librational Dynamic-Response Limits of Gravity-Gradient Satellites

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It is known that the gradient of gravitational-attraction forces generates a restoring couple on a satellite slightly displaced in angular orientation from its equilibrium attitude.[1] The positive restoring moment thus obtained provides a stabilizing influence in purely passive configurations, of interest in such applications as communications reflectors. The weakness of the gravity gradient, however, is equivalent to the use of a very soft spring when displacements are small. This explains excessively long periods of natural oscillation and consequent serious problems of attitude control. Detailed analysis reveals a slight additional stabilizing couple due to inertia and permits appraisal of response characteristics for a wide class of satellite configurations.

An elementary vector identity suggests a compact representation of gravity torque by means of a vector potential

$$\mathbf{W} = -G\int (\mathbf{r}/R)dm$$

where G is the usual product of earth mass times universal gravitational constant, $\mathbf{r}$ is position vector measured from center of moments (which we take as satellite mass center) to a point of the satellite, R is the distance from earth center to the same point, and integration is carried out over all mass points of the satellite. Gravitational torque moment is then given by the vector curl operation as

$$\mathbf{M}_0 = \nabla \times \mathbf{W} \tag{1}$$

where nonzero terms originate only in the term R, which may be written as

$$R^2 = (R_0 + z)^2 + x^2 + y^2$$

R_0 being the distance of satellite mass center from earth center, z is measured outward along the same radial, y is directed oppositely to orbital velocity, and x is such as to form a right-handed Cartesian system. Since R_0 is much greater than x, y, or z, the moment $\mathbf{M}_0$ is given by

$$\mathbf{M}_0 = (3G/R_0^3)[\mathbf{i}\int yzdm - \mathbf{j}\int xzdm] \tag{2}$$

$\mathbf{i}$, $\mathbf{j}$ denoting unit vectors in x, y directions (the same result easily obtains by direct integration of elementary force moments without use of potential $\mathbf{W}$). When principal inertia axes are displaced through infinitesimal angles α, β, γ about x, y, z axes and corresponding inertia moments are denoted by A, B, C, respectively, Eq. (2) is linearly approximated as

$$\mathbf{M}_0 = -(3G/R_0^3)\{\alpha(B - C)\mathbf{i} + \beta(A - C)\mathbf{j}\} \tag{3}$$

With $B > C$, $A > C$, both components have negative signs indicating restoring torques and static stability. These inequalities are appropriate for elongated figures in the radial direction, for which maximum gravity-gradient forces are realized. It is evident that 90° rotations interchange pairs of A, B, C values, and consequent sign reversals indicate instabilities. Expressions (2) or (3) determine the rate of change of total satellite moments of momentum; this has the form

$$\ddot{\alpha}A\mathbf{i}_1 + [\ddot{\beta}B - \dot{\gamma}\Omega B + (\dot{\gamma} + \beta\Omega)\Omega(A - C)]\mathbf{i}_2 + [\ddot{\gamma}C + \dot{\beta}\Omega C - (\dot{\beta} - \gamma\Omega)\Omega(A - B)]\mathbf{i}_3 \tag{4}$$

when referred to principal-axis coordinates. Time derivatives are indicated by dots and orbital angular velocity Ω is introduced, given by

$$\mathbf{\Omega} = \Omega\mathbf{i}_j \quad \Omega^2 = G/R_0^3$$

The separate components of Eqs. (4) reveal dynamic coupling ($\dot{\beta}, \dot{\gamma}$ terms) with orbital motion and centrifugal couples, stabilizing when $A > C$, $A > B$. Within present small angle approximations $\mathbf{i} = \mathbf{i}_1$ and $\mathbf{j} = \mathbf{i}_2$, so that (3) and (4) together determine orientation in space by means of three second-order scalar equations for α, β and γ. Thees equations exhibit the coupling of β and γ motions in all cases except when the condition

$$A - B - C = 0 \tag{5}$$

is satisfied, which is of much less physical interest than the case of the prolate axially symmetric figure for which

$$A = B > C \tag{6}$$

The α motion is in any case uncoupled from the other two modes and is characterized by a natural frequency given by

$$\omega_\alpha = \sqrt{3}\Omega[(B - C)/A]^{1/2} \tag{7}$$

corresponding to less than two cycles per orbit. The maximum value occurs in the limit $C \to 0$, i.e., for configurations everywhere close to the z-axis (for which signal reflection cross-section is also most reduced). From the defining integrals of the moments of inertia it is also seen that the frequency of α motion (oscillations confined to orbital plane) is reduced for the asymmetric case which does not conform to the equality in (6).

The coupled equations for β and γ motion are

$$\begin{aligned} B\ddot{\beta} + \Omega(A - B - C)\dot{\gamma} + 4\Omega^2(A - C)\beta &= 0 \\ C\ddot{\gamma} - \Omega(A - B - C)\dot{\beta} + \Omega^2(A - B)\gamma &= 0, \end{aligned} \tag{8}$$

which bear a close resemblance to those for Foucault's pendulum.[2] This is natural, of course, since our configuration is also a "pendulum," supported by orbital centrifugal force. As in Foucault's case, the two motions are 90° out of phase with each other. Differences appear when the final terms are compared: two distinct frequencies are present in our case if $A \neq B$, and this again signifies reduction in values of natural frequencies. In no case is the system frequency greater than twice the orbital value.

It is therefore seen, even when the centrifugal restoring couple is included, that gravity-gradient passive satellite stability is at best marginal. Large-amplitude displacements are thus invited which lead directly to definite instability. This happens most easily, moreover, for those elongation configurations which give the best response when in their orientation of stable equilibrium. The practical requirement for added system complexity is thus shown.

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